Measuring the Related Properties of Linearity and Elongation of Point Sets

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Abstract. The concept of elongation is generally well understood. However, there is no clear, precise, mathematical definition of elongation in any dictionary we could find. We propose that the definition of elongation should overlap with the definition of linearity since we will show that these two measures produce results that are highly correlated when applied to different types of 2D shapes. Our experiments consist of testing known methods of linearity and elongation on sets of closed shapes contours, shapes whose areas are filled, and shapes with open contours. We tested each algorithm on 25 different shapes in each category. It was found that the Average Orientations linearity measure from [10] best correlates to the elongation measures found in literature. It has a correlation value of above 0.9 with measures of elongation for open and closed curves. Also, we have discovered that the standard measure of elongation, applied to its intended area based shapes, gives almost identical results when it is applied to just the boundary pixels of the same area based shapes. They are over .98 correlated. This leads to a new linearity/elongation measure which is fast, applicable to both open and closed shapes, is given by a closed formula, and highly agrees with existing measures.

Keywords: Linearity, elongation, unordered point sets.

1 Introduction

The elongation of an object is understood to be something that gives an idea of the length vs. the width of that object. The Webster's dictionary definition of this term articulates that elongation is 'the quality of being elongated'. A further search of the term 'elongated' gives 'having notably more length than width'. This is hardly a concise definition, so we present our own set of definitions to accurately define the term 'elongation'. Measuring elongation of a finite set of points in 2d space is equivalent to measuring the linearity of the same set. The linearity of a point set indicates how close this set is to a straight line. We have compared methods measuring linearity with methods of measuring elongation in literature, against the same test set. By correlating the results of both approaches, we were able to empirically show a strong correlation between the two ways of measuring what appears to be the same thing. Elongation/linearity is a useful tool in shape classification tasks in image processing, which is why we devote this article to studying it further.

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In considering various linearity and elongation algorithms, we align ourselves with the following criteria. We are interested in those that assign values to sets of points in the range [0, 1]. They are equal to 1 if and only if the shape is perfectly linear or elongated, and equals 0 when the shape is highly circular or has another form which is highly non-linear. A shape's linearity and elongation value should be invariant under similarity transformations of the shape, such as scaling, rotation and translation. The algorithms should also be resistant to protrusions in the data set. Linearity and elongation values should also be computed by simple and fast algorithms.

Elongation methods in literature typically yield results in the interval [1, ∞). These measures are transferred to the interval [0, 1] by the following calculation. If elongation value e in the range [1, ∞), then it is equal to 1-1/e in the range [0, 1].

It is important to stress that points in the sets we are considering are not ordered. This means that figures such as ellipses or rectangles which are very flat (long and thin) are considered to be highly linear, and therefore also highly elongated. If we were to consider ordered sets of points, such ellipses would be highly nonlinear. It is also impossible to select a consistent ordering of points in shapes which include large areas of pixels, such as filled circles. It is for this reason that we chose to apply both linearity and elongation methods to unordered point sets.

Here, we consider 5 methods of finding linearity and 5 methods of finding elongation. The linearity algorithms are taken from [10]. Three of the elongation measures are taken from [11] and one from [12]. The fifth measure of elongation is the standard area based method proposed in [2]. The 'eccentricity' measure from [12] was actually at one point used as a linearity measure in [10]. Here it is once again considered an elongation measure. The algorithms are sensitive to large extrusions in the curve but they mainly do not react to small ones which could be due to noise.

There are many publications that deal with elongation: [2, 3, 4]. The standard measure of shape elongation is derived from the definition of shape orientation that is based on the axis of the least second moment of inertia. Precisely, the axis of the least second moment of inertia ([2, 3, 4]) is the line which minimizes the sum of the squares of distances of the points (belonging to the shape) to the line.

The literature review is given in section 2. The test sets and comparison procedures are outlined in section 3. The results of the algorithms which were tested on various curves, are presented in section 4 along with a discussion of the results.

2 Literature Review

We will describe several well known functions on finite sets of points that are used in our linearity measures here. Existing linearity measures for unordered set will be covered along with other relevant measures.

2.1 Linearity Measures for Unordered Sets

The most relevant and applicable shape measure to our work is the measuring of linearity of unordered data sets. [10] is the only source in literature that deals directly with measuring linearity for unordered sets of points. Five linearity measures were proposed in [10], all of which we will use here. The *average orientation* (AO) scheme first finds the orientation line of the set of points using moments. The method takes *k* pairs of points and finds the unit normals to the lines that they form. The unit normals all point in the same direction (along the normal to the orientation line). The average normal value (*A*, *B*) of all of the *k* pairs is found, and the linearity value is calculated as $\sqrt{A^2 + B^2}$. *Triangle heights* (TH) takes an average value of the relative heights of triangles formed by taking random triplets of points. Relative heights are heights that are divided by the longest side of the triangle, then normalized so that we obtain a linearity value in the interval [0, 1]. *Triangle perimeters* (TP) takes the normalized, average value of the area divided by the square of the perimeter of triplets of points as its linearity measure. *Contour smoothness* (CS) was adapted from a measure of circularity. It is a simple formula involving moments that were found in literature, and adapted to finding linearity [12]. The idea remained the same, but the resulting measurements were interpreted differently. In the original scheme in [12], they proposed a measure of circularity by dividing the area of a shape by the square of its perimeter. For circles, they arrived at circularities of 1, and values of less than 1 for other objects. *Ellipse axis ratio* (EAR) is based on the minor/major axis ratio of the best ellipse that fits the set of points.

2.2 Elongation Measures

Eccentricity (E) was the simplest measure of elongation we could find. It was also used in [12]. The output of this algorithm is already in the interval [0, 1], so there was no need to normalize it. For a disc, this measure outputs 0, for a line, it outputs 1 since lines are eccentric. The formula is

$$E = \frac{\sqrt{(\mu_{20} - \mu_{02})^2 + 4\mu_{11}^2}}{\mu_{20} + \mu_{02}}.$$

The *standard measure* of shape elongation is derived from the definition of shape orientation that is based on the axis of the least second moment of inertia. The minimum and maximum sums of projection edges for the shape are computed as follows:

$$\max = \frac{\mu_{20} + \mu_{02} + \sqrt{4 \cdot (\mu_{11})^2 + (\mu_{20} - \mu_{02})^2}}{2}$$

and

$$\min = \frac{\mu_{20} + \mu_{02} - \sqrt{4 \cdot (\mu_{11})^2 + (\mu_{20} - \mu_{02})^2}}{2}.$$

The elongation of the given shape is defined as the *max-to-min ratio* (MMR), where *MMR* = max/min. The MMR is the standard measure of elongation of a given shape. Some generalization of the standard method for measuring shape elongation can be found in [13]. The standard measure (MMR) of shape elongation is area based because all pixels belonging to the shape are involved in the computation (area moments are used).

Let *P* be a shape with a polygonal boundary. An elongation measure is defined in [11] as the ratio of the maximum and minimum value over candidate straight lines of the function

$$\sum_{e \text{ is an edge of } P} |pr_a(e)|^2 .$$

The elongation measure of *P* can be expressed as [11]:

$$SZ = \frac{\sum_{1\le i\le n} |e_i|^2 + \sqrt{\left(\sum_{1\le i\le n} |e_i|^2 \cos(2\alpha_i)\right)^2 + \left(\sum_{1\le i\le n} |e_i|^2 \sin(2\alpha_i)\right)^2}}{\sum_{1\le i\le n} |e_i|^2 - \sqrt{\left(\sum_{1\le i\le n} |e_i|^2 \cos(2\alpha_i)\right)^2 + \left(\sum_{1\le i\le n} |e_i|^2 \sin(2\alpha_i)\right)^2}},$$

where e_i $(1 \le i \le n)$ are edges of the boundary of *P*, $pr_a(e)$ is the projection of edge *e* along the line *a*, and α_i $(1 \le i \le n)$ are angles between the edges e_i and the *x*-axis. Note that this measure is used for polygonal shapes only. However, it can be applied to arbitrary shapes by considering line segments between consecutive pixels as edges of a polygon.

The elongation measure for polygon *P* [14] is the ratio of the maximum and minimum value of

$$\sum_{e \text{ is an edge of } P} |pr_a(e)|^2 \big/ |e| ,$$

and is equal to

$$ZS = \frac{\sum_{1\le i\le n} |e_i| + \sqrt{\left(\sum_{1\le i\le n} |e_i| \cos(2\alpha_i)\right)^2 + \left(\sum_{1\le i\le n} |e_i| \sin(2\alpha_i)\right)^2}}{\sum_{1\le i\le n} |e_i| - \sqrt{\left(\sum_{1\le i\le n} |e_i| \cos(2\alpha_i)\right)^2 + \left(\sum_{1\le i\le n} |e_i| \sin(2\alpha_i)\right)^2}}.$$

The final measure of shape elongation we consider here extends the polygonal elongation measure to shapes with arbitrary boundaries. Assume that we have a piecewise smooth enough curve *P* given in a parametric form $x = x(t),\ y = y(t),\ (t \in [a, b])$. The elongation ZSC of the curve *P* is defined as [14]:

$$ZSC = Length(P) + \sqrt{\left(\int_a^b \frac{2\dot{x}\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}dt\right)^2 + \left(\int_a^b \frac{\dot{x}^2-\dot{y}^2}{\sqrt{\dot{x}^2+\dot{y}^2}}dt\right)^2} \Bigg/ Length(P) - \sqrt{\left(\int_a^b \frac{2\dot{x}\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}dt\right)^2 + \left(\int_a^b \frac{\dot{x}^2-\dot{y}^2}{\sqrt{\dot{x}^2+\dot{y}^2}}dt\right)^2}$$

where $\dot{x} = dx/dt$ and $\dot{y} = dy/dt$. Note that the MMR, 11 and SZC measures are defined in the interval $[1, \infty)$, and were converted to the interval $[0, 1]$ before correlating them with the other measures.

[14] have shown that the measure ZSC satisfies the “convergence property”. Precisely, let us assume we have a curve and a set of sample points from it. Also, let us assume that we have the computed elongation ZS of the polygonal curve *P* whose vertices are the selected sample points. Then, roughly speaking, by the convergence property of an elongation measure, the computed elongations ZS (of polygonal curves determined by sample points) should converge towards ZSC when the density of sample points increases and the largest distance between any two consecutive sample

points approaches zero. Naturally, if the convergence property holds, the limit value for the measured elongations ZS of polygonal lines determined by sample points is used as the elongation measure *ZSC* of the sampled curve. Our tables give the results of the ZS measure with edges composed of consecutive shape pixels, as an approximation for the ZSC measure of continuous curves.

Let us mention that there are also some naive measures of elongation. For example, shape elongation can be measured as the ratio of the longer and shorter edges of the minimum area bounding rectangle for the measured shape. It is worth mentioning that such bounding rectangles are easy to compute [1, 5]. These measures are area based and are sensitive to protrusions. They were not included in the experimentation.

3 Measuring Linearity and Elongation

Here, we will describe the test methodology of the linearity and elongation measures and present the test sets of shapes. The test sets are shown below. Figure 1 shows the set of closed shape contours, most of which were taken from the test sets used in [6]. Figure 2 shows the area based curves test set which was mostly taken from [8]. Figure 3 shows the open shape contours test set which was partly taken from [9]. There are 23 closed shape contours, 25 area based shapes and 25 open shape contours. Each contour based shape has between 300 and 800 pixels. The area based shapes have roughly the same number of boundary pixels, but since their areas are also considered, their pixel count is significantly higher.

It was discovered that all of the algorithms for linearity and elongation can be applied to each test set. The standard elongation measure is area based and thought to be only applicable to area based shapes. However, we have shown that even the standard MMR measure can be applied to boundary based shapes, by applying the same formula on just the boundary pixels rather than all of the pixels inside a shape. The new measure will be referred to as Standard Boundary (SB).

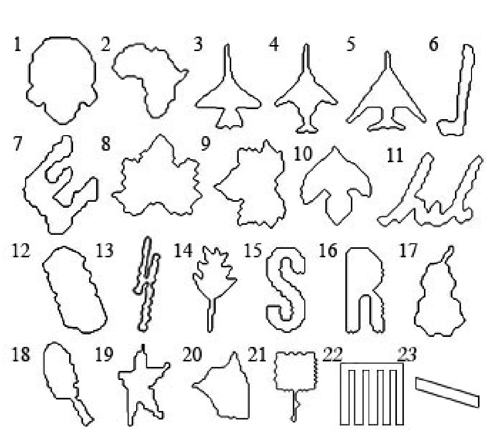

Fig. 1. Closed shape contours

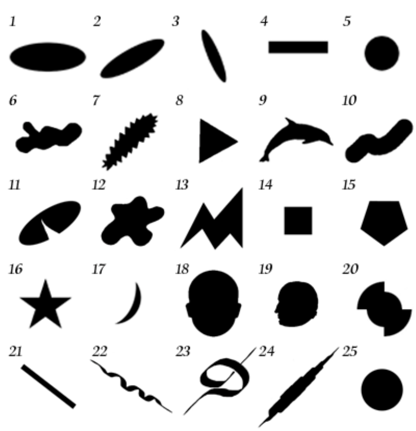

Fig. 2. Area based shapes

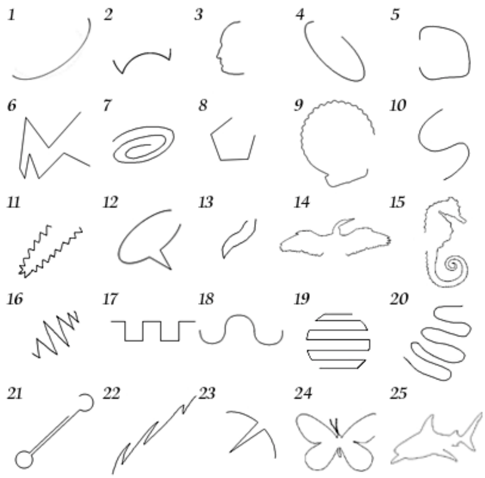

Fig. 3. Open shape contours

4 Experimental Data and Results

Table 1 shows the linearity and elongation results for the closed shape contours seen in Figure 1.

Table 1. Linearity and Elongation for Closed shape contours

	AO	TH	TP	CS	EAR	E	SB	SZ	ZS
1	.14	.13	.16	.11	.14	.14	.26	.18	.15
2	.27	.13	.14	.12	.25	.28	.43	.06	.05
3	.23	.17	.24	.13	.38	.45	.62	.14	.12
4	.31	.19	.25	.16	.46	.55	.71	.27	.22
5	.06	.08	.10	.06	.14	.15	.27	.08	.08
6	.71	.53	.65	.45	.76	.89	.94	.64	.60
7	.16	.01	.01	.01	.24	.26	.42	.44	.48
8	.11	.07	.07	.06	.13	.14	.25	.04	.04
9	.07	.05	.06	.03	.23	.26	.41	.22	.18
10	.09	.03	.05	.02	.05	.05	.12	.27	.22
11	.37	.12	.17	.10	.34	.40	.57	.56	.58
12	.33	.16	.22	.12	.41	.49	.66	.32	.34
13	.65	.41	.52	.34	.78	.91	.95	.68	.64
14	.41	.29	.37	.23	.55	.66	.80	.41	.36
15	.42	.18	.25	.14	.54	.64	.78	.34	.36
16	.44	.21	.29	.16	.47	.56	.72	.62	.56
17	.44	.20	.26	.16	.46	.55	.71	.36	.32
18	.57	.38	.51	.30	.62	.75	.86	.51	.53
19	.30	.10	.13	.09	.40	.47	.64	.39	.36
20	.12	.01	.00	.01	.16	.17	.29	.22	.21
21	.31	.16	.23	.12	.35	.41	.58	.29	.26
22	.08	.04	.01	.04	.01	.01	.01	.78	.78
23	.73	.66	.79	.59	.87	.97	.98	.76	.77

Table 2. Linearity and elongation for area based shapes

	AO	TH	TP	CS	EAR	E	MMR	SB	SZ	ZS
1	.39	.16	.21	.13	.51	.62	.85	.76	.68	.62
2	.45	.28	.35	.25	.65	.78	.92	.88	.74	.76
3	.60	.37	.46	.31	.72	.86	.96	.92	.77	.78
4	.63	.38	.47	.32	.72	.86	.96	.92	.80	.80
5	.08	.15	.19	.12	.00	.00	.00	.00	.01	.00
6	.45	.19	.24	.16	.54	.65	.87	.79	.41	.36
7	.54	.29	.37	.25	.64	.77	.92	.87	.19	.17
8	.03	.10	.09	.10	.01	.01	.00	.02	.15	.13
9	.40	.34	.44	.26	.55	.66	.85	.79	.40	.36
10	.47	.29	.37	.24	.61	.74	.91	.85	.58	.61
11	.39	.19	.27	.12	.53	.64	.86	.78	.34	.37
12	.19	.10	.10	.10	.26	.29	.50	.45	.28	.29
13	.25	.07	.10	.05	.33	.38	.57	.55	.62	.58
14	.06	.02	.01	.04	.00	.00	.00	.00	.01	.01
15	.09	.10	.14	.06	.01	.01	.00	.01	.05	.04
16	.00	.08	.06	.07	.01	.01	.01	.02	.05	.04
17	.71	.57	.73	.47	.71	.85	.93	.91	.62	.64
18	.14	.08	.10	.05	.13	.14	.37	.24	.17	.15
19	.11	.05	.07	.03	.11	.12	.25	.22	.17	.20
20	.14	.09	.13	.06	.25	.28	.62	.44	.14	.17
21	.79	.69	.80	.63	.87	.97	.99	.98	.84	.86
22	.85	.74	.86	.67	.91	.98	.99	.99	.66	.71
23	.39	.15	.23	.12	.51	.61	.61	.76	.55	.57
24	.84	.63	.78	.55	.87	.97	.98	.98	.91	.93
25	.04	.09	.12	.05	.00	.00	.00	.00	.01	.01

Table 2 shows the linearity and elongation results for the area based shapes seen in Figure 2. Table 3 shows the linearity and elongation results for the open shape contours seen in Figure 3. The first 5 columns of each of the three tables list the linearity results of each shape. They are: Average Orientations (AO), Triangle Heights (TH), Triangle Perimeters (TP), Contour Smoothness (CS), and Ellipse Axis Ratio (EAR). The Elongation measures are listed in the last four columns, and they are: Eccentricity (E), the standard measure of elongation as referred to in [13] (MMR), our new measure *standard boundary* SB which is an elongation measure for just the boundary points of the area shapes as calculated by the same formula as MMR, the first method of elongation defined in [11] (SZ), and finally the convergent method of elongation as defined in [14] (ZS).

In order to compare our results, we correlate the relevant columns in each table to see if elongation and linearity are related. In each table we see the linearity algorithms

Table 3. Linearity and elongation for open shape contours

	AO	TH	TP	CS	EAR	E	SB	SZ	ZS
1	.85	.78	.93	.70	.80	.92	.96	.59	.65
2	.70	.67	.81	.59	.80	.92	.96	.16	.14
3	.53	.54	.70	.43	.61	.74	.85	.29	.32
4	.50	.33	.40	.27	.57	.69	.82	.64	.70
5	.09	.17	.19	.14	.21	.23	.31	.28	.30
6	.05	.12	.15	.11	.16	.17	.30	.48	.41
7	.38	.13	.21	.09	.40	.46	.64	.67	.65
8	.07	.21	.25	.17	.25	.28	.44	.32	.37
9	.00	.20	.26	.16	.23	.26	.40	.15	.16
10	.19	.15	.19	.11	.44	.52	.68	.50	.54
11	.46	.27	.34	.23	.56	.67	.82	.26	.23
12	.25	.07	.07	.07	.17	.18	.33	.37	.40
13	.53	.29	.39	.23	.67	.81	.89	.74	.79
14	.62	.37	.52	.28	.68	.81	.90	.42	.37
15	.38	.27	.35	.22	.57	.69	.82	.13	.13
16	.39	.21	.30	.16	.53	.64	.77	.71	.76
17	.50	.31	.41	.24	.65	.78	.88	.08	.08
18	.48	.39	.53	.29	.60	.73	.84	.05	.06
19	.04	.05	.03	.05	.07	.07	.16	.88	.86
20	.21	.04	.08	.04	.32	.37	.54	.65	.65
21	.79	.73	.84	.67	.87	.97	.98	.72	.76
22	.84	.73	.84	.66	.90	.98	.99	.86	.89
23	.07	.13	.16	.12	.27	.31	.46	.44	.47
24	.18	.02	.03	.02	.25	.28	.43	.15	.16
25	.45	.22	.29	.18	.56	.67	.80	.34	.36

listed as the columns and the elongation measures listed as the rows. Each cell represents the correlation value between the measures of the corresponding linearity and elongation for a set of curves.

Table 4 shows the correlation values for the area based shapes seen in Figure 2. Here we see that the correlation values are all very high in each cell. The AO and EAR methods best correlate to the elongation schemes of E, MMR, and SB. We notice that the MMR and SB methods have nearly identical correlation values with each of the linearity measures in Table 4. We further examine the relationship between the MMR elongation measure for area and boundary shapes by correlating their results. It was found that these two measures have a correlation factor of .989. This is strong evidence that the area based measure, and the moment functions that it relies on can be used on boundary shapes as well. For this reason, the MMR measure was also compared to the open and closed shapes in Figures 1 and 3.

Table 4. Correlations for area based shapes

	AO	TH	TP	CS	EAR
E	0.966	0.826	0.853	0.810	0.998
MMR	0.898	0.728	0.761	0.707	0.961
SB	0.924	0.743	0.777	0.724	0.980

Table 5 shows the correlation results of the linearity and elongation measures on open and closed curves from Figures 1 and 3. We immediately notice that the correlation values are quite high for the Eccentricity and MMR elongation measures compared to all of the linearity measures for both open and closed shapes. The SZ and ZS measures for closed curves are not correlated as highly to the linearity values for closed shapes, and it are not correlated at all with the linearity values for open shapes.

Table 5. Correlations for open and closed curves

		AO	TH	TP	CS	EAR
Closed	**E**	0.960	0.898	0.921	0.875	0.998
	SZ	0.638	0.574	0.557	0.584	0.549
	ZS	0.621	0.552	0.533	0.563	0.529
	SB	0.920	0.819	0.854	0.790	0.976
Open	**E**	0.946	0.842	0.874	0.813	0.996
	SZ	0.186	0.097	0.066	0.139	0.098
	ZS	0.199	0.121	0.088	0.162	0.119
	SB	0.907	0.779	0.820	0.742	0.978

The reason for such low correlation values between the linearity measures and the elongation of open shapes as calculated by SZ and ZS lies in the way elongation is calculated in for these measures. Consider Figure 4 for clarification. There we see three shapes that would have the same elongation value according to the non convergent definition of elongation defined in [11]. This happens since their elongation definition is the ratio of horizontal and vertical edges projected onto the x and y axes. We see that since the shapes a, b, and c in Figure 4 have the same ratio of horizontal and vertical edges, their elongation value would also be the same. Shapes 4, 5, 6, 7, 9, 10, 11, 16, 17, 18, 19, 20 and 23 in Figure 3 all exhibit properties such as those seen in Figure 4. Their elongation values as calculated by SZ and ZS are consequently much lower than they should be, and therefore when the set of results is correlated with the linearity measures, no direct link can be seen. This is not a reflection of a non existent link between elongation and linearity, it is a testament of situations where the elongation measure of [11, 14] does not perform adequately.

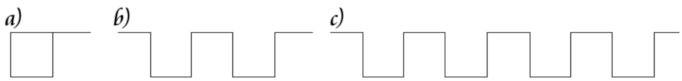

Fig. 4. Three shapes having the same elongation according to [11]

A two tailed, paired t test of the MMR and SB measures on the area based shape set yields a value of 0.0161. Their mean difference of measures is 0.03096, which means that on average, the measures produce results which vary by 3%. A confidence interval of 95% specifies that the measures will produce values that will differ in the range [0.00627, 0.05565].

5 Conclusion

We have seen that the measures of elongation and linearity are highly correlated on various sets of data. This can lead us to conclude that these measures are relatively interchangeable, if not completely equivalent. Further experimentation should be done on real world data that actively interchanges these measures in order to be able to better support our assertions. We have also concluded that the MMR measure can be applied equally to area and shape based figures. This leads us to believe that calculations involving moments of inertia are not strictly limited to area based shapes. A formal proof is missing for this claim, and is left as future work.

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